Let $(a_n)$ be a bounded sequence of real numbers. Let us remind the notations

$$R_n := \{a_k \,:\, k \geq n\}\,, \quad s_n := \sup R_n, \quad m_n := \inf R_n.$$

Let us also recall the following definition:

**Definition.** Limit superior of $(a_n)$ is defined as

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} s_n = \inf s_n.$$

Limit inferior of $(a_n)$ is defined as

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} m_n = \sup m_n.$$

**Theorem.** *Let $(a_n)$ be a bounded sequence of real numbers. Then*

*(1) For some subsequence $(a_{n_k})$,*

$$\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n.$$

*(2) For some subsequence $(a_{n_k})$,*

$$\lim_{k\to\infty} a_{n_k} = \liminf_{n\to\infty} a_n.$$

*(3) If for some subsequence $(a_{n_k})$,*

$$a = \lim_{k\to\infty} a_{n_k}$$

*then*

$$\liminf_{n\to\infty} a_n \leq a \leq \limsup_{n\to\infty} a_n.$$

*Proof.* To prove 1, let us construct recursively a sequence $n_1 < n_2 < \dots n_k < \dots$ such that

(*) $$a_{n_k} > s_{n_{k-1}+1} - 2^{-k}.$$

Then, since $a_{n_k} \in R_{n_{k-1}+1}$, we have

$$a_{n_k} \leq s_{n_{k-1}+1}.$$

Observe that

$$\lim_{k\to\infty} s_{n_{k-1}+1} = \lim_{k\to\infty} s_{n_{k-1}+1} - 2^{-k} = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} a_n$$

Thus, by the Squeezed sequence lemma (applied to $s_{n_{k-1}+1}$ and $s_{n_{k-1}+1} - 2^{-k}$)

To construct a subsequence with the property (*), let $n_1$ be an index such that

$$a_{n_1} > s_1 - 2^{-1}.$$

Such an index exists since $s_1 = \sup R_1$.

If $n_{k-1}$ is already constructed, let $a_{n_k}$ be an element of $R_{n_{k-1}+1}$ with $a_{n_k} > s_{n_{k-1}+1} - 2^{-k}$. Again, it exists since $s_{n_{k-1}+1} = \sup R_{n_{k-1}+1}$. Also, $n_k \geq n_{k-1} + 1 > n_{k-1}$.

The proof of 2 is the same as the proof of 1.

To prove 3, let us observe that $m_{n_k} \leq a_{n_k} \leq s_{n_k}$, and both sequences $(m_{n_k})$ and $(s_{n_k})$ converge to $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ respectively, as subsequences of convergent sequences $(m_n)$ and $(s_n)$. Thus

$$\liminf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$

□

**Corollary** (Bolzano-Weierstrass Theorem). *Let $(a_n)$ be a bounded sequence of real numbers. Then it has a convergent subsequence.*

*Proof.* Take a subsequence converging to $\limsup_{n\to\infty} a_n$. $\square$

**Corollary.** *Let $(a_n)$ be a bounded sequence of real numbers and $a \in \mathbb{R}$. Assume that every* convergent *subsequence of $(a_n)$ converges to $a$. Then $(a_n)$ itself converges to $a$.*

*Proof.* There is a subsequence of $(a_n)$ convergent to $\limsup_{n\to\infty} a_n$. Thus $\limsup_{n\to\infty} a_n = a$. By the same reasoning, $\liminf_{n\to\infty} a_n = a$. Therefore,

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a,$$

and thus

$$\lim_{n\to\infty} a_n = a.$$

$\square$